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ON THE DEGENERATE LAPLACE TRANSFORM - III

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Abstract

Proceeding ahead in our earlier studies [31, 32] which are in progression of the very recent study of Kim and Kim [30], in this report we give an expression for the degenerate Laplace transform of the degenerate Laplace transform of a function and also define the degenerate sine integral function and the degenerate cosine integral function and obtain the expressions for the degenerate Laplace transforms of these two functions.

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1. Introduction

Owing to the very deep applications of the classical Laplace transform in engineering problems, many researchers have focused their attention to the studies of this integral transform. Many generalizations of the classical Laplace transforms are available in the literature. The most recent of such generalizations is the very fruitful concept of the *degenerate Laplace transform* introduced by Kim and Kim in their novel work [30]. A number of problems in applied mathematics, physics, electrical and electronics engineering often depend upon the solutions of differential equations which are usually solved in these fields by the applications of the classical Laplace transform. The solutions of such differential equations in advanced and complex problems of engineering, physics and applied mathematics are often described in terms of a number of special functions of mathematics which often require the knowledge of the Laplace transforms of these special functions. Many authoritative and monumental works are written in mathematics

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which deal with the theory and applications of the numerous special functions of mathematics and the Laplace transforms of these functions, such as [1 - 12] besides numerous others. Motivated by the fact that the study of the classical Laplace transform and its many varied applications form a core part of the syllabi of mathematics courses of trainee engineers, physicists and mathematicians worldwide we propose to utilize the results of the very recent work of Kim and Kim [30] on the degenerate Laplace transforms to find the expressions for the degenerate Laplace transform of the degenerate Laplace transform of a function besides defining the *degenerate sine integral function* and the *degenerate cosine integral function* and obtain their degenerate Laplace transforms. We would like to mention that prior to giving the most recent concept of the degenerate Laplace transform [30], T .Kim and his fellow research collaborators D.V. Dolgy, J.J. Seo, L.C. Jang, H.-I. Kwon and D.S. Kim have made numerous studies on the degenerate polynomials and the degenerate numbers [22-29]. Prior to them, Carlitz [13, 14] also studied the degenerate analogues of numbers and functions. Mathai [21] has done a commendable work by using the Laplace transform approach to study the special functions of matrix arguments with real symmetric positive definite matrices and the Hermitian positive definite matrices as arguments. Earlier this author has also given the generalizations of the Laplace type integrals of various multiple hypergeometric functions with real symmetric positive definite matrices as their arguments in his studies [for instance see, 16-19] and his doctoral dissertation [20] and many of his other previous studies which are mentioned in the references of his doctoral dissertation [20].

The scheme of the paper is as follows: we divide the paper in two sections, the first section details the necessary definitions and results which are used to derive the main results of this investigation in the second section of the paper. We begin to enumerate below the basic definitions and results. The Laplace transform of a function $f(t)$ of the variable t defined for $t>0$, denoted by $\mathcal{L}\{f(t)\}$, is defined by the integral (see, for instance, [15, (1), p.1])

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{1.1}$$

if the integral in (1.1) converges for some value of the complex parameter s. For sufficient conditions for the existence of the Laplace transform of a function $f(t)$, the reader is referred to the Theorem 1-1, p.2 and the Problem 145, p.38 of [15].

Definition 1.1: The Degenerate Exponential Function - (Kim and Kim [30], (1.3), p. 241) – The degenerate exponential function, represented by e_λ^t, is a function of two variables λ and t, where, $\lambda \in (0,\infty)$, $t \in \mathbb{R}$ and is defined by

$$e_\lambda^t = (1+\lambda t)^{\frac{1}{\lambda}} \tag{1.2}$$

We remark here that this definition generalizes the classical exponential function e^t defined by the well known series relation

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{1.3}$$

because we can easily deduce from (1.2) that (see Kim and Kim [30], p.241)

$$\lim_{\lambda\to 0+} e_\lambda^t = \lim_{\lambda\to 0+} (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t \tag{1.4}$$

The well known Euler's exponential formula is given by (see, for instance, (1.7) p.241 Kim and Kim [30])

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.5}$$

where $i=\sqrt{-1}$. From (1.5) the definitions of the elementary trigonometric functions sine and cosine in terms of the exponential function as (see, for instance, (1.8) p.241 Kim and Kim[30]) immediately follow as

$$\cos a\theta = \frac{e^{ia\theta}+e^{-ia\theta}}{2}, \quad \sin a\theta = \frac{e^{ia\theta}-e^{-ia\theta}}{2i} \tag{1.6}$$

Definition 1.2: The Degenerate Euler Formula: The degenerate Euler formula is defined by the relation (see (1.9), p.242 Kim and Kim [30])

$$e_\lambda^{it} = (1+\lambda t)^{\frac{i}{\lambda}} = \cos_\lambda(t) + i\sin_\lambda(t) \tag{1.7}$$

The limiting case of (1.7) easily leads to the conclusion that (see (1.10) p.242 Kim and Kim [30])

$$\lim_{\lambda\to 0+} e_\lambda^{it} = \lim_{\lambda\to 0+}(1+\lambda t)^{\frac{i}{\lambda}} = e^{it} = \cos t + i\sin t \tag{1.8}$$

Further, from (1.7) and (1.8) one can arrive at the following deductions without any difficulty (see (1.11) p.242 Kim and Kim [30])

$$\lim_{\lambda\to 0+}\cos_\lambda(t) = \cos t, \quad \lim_{\lambda\to 0+}\sin_\lambda(t) = \sin t \tag{1.9}$$

Definition 1.3: The Degenerate Cosine and Degenerate Sine Functions: The following respective definitions of the degenerate cosine and degenerate sine functions follow from (1.7) (see (1.12), p.242 Kim and Kim[30])

$$\cos_\lambda(t) = \frac{e_\lambda^{it}+e_\lambda^{-it}}{2}, \quad \sin_\lambda(t) = \frac{e_\lambda^{it}-e_\lambda^{-it}}{2i} \tag{1.10}$$

Definition 1.4: The Degenerate Laplace Transform of a Function: (Kim and Kim [30], (3.1), p.244) Let $f(t)$ be a function defined for $t\geq 0$ and let $\lambda\in(0,\infty)$, then the degenerate Laplace transform of the function $f(t)$, represented by $F_\lambda(s)$, is defined by the integral

$$\mathcal{L}_\lambda\{f(t)\} = F_\lambda(s) = \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)\,dt \tag{1.11}$$

2. Results for The Degenerate Laplace Transforms of Some Functions

We embark upon our aim of this study now as set out in the abstract as well as in the first section of this paper above. First we deduce an expression for the degenerate Laplace transform of the degenerate Laplace transform of a function which we shall state in the form of a theorem and then define the degenerate sine integral and the degenerate cosine integral and find their degenerate Laplace transforms.

Theorem 2.1: The Degenerate Laplace Transform of the Degenerate Laplace Transform of a Function: Let $\mathcal{L}_\lambda\{f(t)\}=F_\lambda(s)$ then we have

$$\mathcal{L}_\lambda\{\mathcal{L}_\lambda\{f(t)\}\}$$

$$=\frac{-1}{\lambda-u}\int_0^\infty f(t)\,dt + \frac{1}{\lambda(\lambda-u)}\int_0^\infty f(t)\log(1+\lambda t)\left[\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}(1+\lambda s)^{\frac{\lambda-u}{\lambda}}\,ds\right]dt \tag{2.1}$$

$$= \quad \lambda\int_0^\infty \frac{f(t)}{\log(1+\lambda t)}dt \; - \; u\lambda\int_0^\infty \frac{f(t)}{\log(1+\lambda t)}\left[\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}(1+\lambda s)^{\frac{-(u+\lambda)}{\lambda}}ds\right]dt \quad (2.2)$$

where u is the parameter of the outer degenerate Laplace transform operator in the expression $\mathcal{L}_\lambda\{\mathcal{L}_\lambda\{f(t)\}\}$.

Proof: Applying the definition of the degenerate Laplace transform, i.e. (1.11) we observe that

$$\mathcal{L}_\lambda\{\mathcal{L}_\lambda\{f(t)\}\} = \mathcal{L}_\lambda\left\{\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)dt\right\}$$

$$= \int_0^\infty (1+\lambda s)^{\frac{-u}{\lambda}} ds\left[\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)dt\right] \quad (2.3)$$

It may be noted here that the area of integration in the double integral of (2.3) is the entire positive quadrant and both the variables s and t are independent therefore, on changing the order of integration we have

$$\mathcal{L}_\lambda\{\mathcal{L}_\lambda\{f(t)\}\} = \int_0^\infty f(t)dt\left[\int_0^\infty (1+\lambda s)^{\frac{-u}{\lambda}}(1+\lambda t)^{\frac{-s}{\lambda}} ds\right] \quad (2.4)$$

Let

$$I = \int_0^\infty (1+\lambda s)^{\frac{-u}{\lambda}}(1+\lambda t)^{\frac{-s}{\lambda}} ds \quad (2.5)$$

This integral can be evaluated by the application of integration by parts in two ways by taking the functions $(1+\lambda s)^{\frac{-u}{\lambda}}$ and $(1+\lambda t)^{\frac{-s}{\lambda}}$ successively as the first functions. When the function $(1+\lambda t)^{\frac{-s}{\lambda}}$ is taken as the first function, then the evaluation of (2.5) can be done as below:

$$I = \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}(1+\lambda s)^{\frac{-u}{\lambda}} ds$$

$$= \left[(1+\lambda t)^{\frac{-s}{\lambda}}\frac{(1+\lambda s)^{\frac{\lambda-u}{\lambda}}}{\lambda-u}\right]_{s=0}^\infty - \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}\log(1+\lambda t)^{\frac{-1}{\lambda}}\frac{(1+\lambda s)^{\frac{\lambda-u}{\lambda}}}{\lambda-u}ds$$

$$= \quad 0 \; - \; \frac{1}{\lambda-u} + \frac{1}{\lambda(\lambda-u)}\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}\log(1+\lambda t)(1+\lambda s)^{\frac{\lambda-u}{\lambda}}ds \quad (2.6)$$

On the other hand when we take the function $(1+\lambda s)^{\frac{-u}{\lambda}}$ as the first function, the evaluation of (2.5) is as follows

$$I = \int_0^\infty (1+\lambda s)^{\frac{-u}{\lambda}}(1+\lambda t)^{\frac{-s}{\lambda}} ds$$

$$= \left[(1+\lambda s)^{\frac{-u}{\lambda}}\frac{(1+\lambda t)^{\frac{-s}{\lambda}}}{\log(1+\lambda t)^{\frac{-1}{\lambda}}}\right]_{s=0}^\infty + u\int_0^\infty \frac{(1+\lambda t)^{\frac{-s}{\lambda}}}{\log(1+\lambda t)^{\frac{-1}{\lambda}}}(1+\lambda s)^{\frac{-u}{\lambda}-1}ds$$

Or,

$$I = \frac{\lambda}{\log(1+\lambda t)} - u\lambda\int_0^\infty \frac{(1+\lambda t)^{\frac{-s}{\lambda}}}{\log(1+\lambda t)}(1+\lambda s)^{\frac{-u}{\lambda}-1}ds \quad (2.7)$$

On substituting the value of I from (2.6) into (2.4), the result in (2.1) follows. Similarly, using the value of I from (2.7) in (2.4) produces the result of (2.2).

Definition 2.2: The Degenerate Sine Integral: We define the *degenerate sine integral* function by the integral

$$\mathrm{Si}_\lambda(at)=\int_0^t \frac{\lambda \sin_\lambda(au)}{\log(1+\lambda u)}du \tag{2.8}$$

for any $a\in\mathbb{C}$. It can be seen that for $a=1$ and as $\lambda\to 0+$ in (2.8) then it reduces to the usual sine integral function (see, for instance, (8), Appendix C, p.255, [15]) given by

$$\mathrm{Si}(t)=\int_0^t \frac{\sin u}{u}du \tag{2.9}$$

We now give in the theorem below an expression for the degenerate Laplace transform of the degenerate sine integral function:

Theorem 2.3: The Degenerate Laplace Transform of the Degenerate Sine Integral:

$$\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\mathrm{Si}_\lambda(at)\right\}=\frac{1}{s}\tan^{-1}\left(\frac{a}{s-\lambda}\right) \tag{2.10}$$

Proof: Kim and Kim [30]have proved that (see (3.9), p.245 Kim and Kim [30])

$$\mathcal{L}_\lambda\left\{\sin_\lambda(at)\right\}=\frac{a}{(s-\lambda)^2+a^2} \tag{2.11}$$

We have proved in our previous study [32] (see Theorem (2.2), p.66 of [32]) the division theorem of the degenerate Laplace transform as below:

If $\mathcal{L}_\lambda\{f(t)\}=F_\lambda(s)$ then

$$\mathcal{L}_\lambda\left\{\frac{\lambda f(t)}{\log(1+\lambda t)}\right\}=\int_s^\infty F_\lambda(s)ds \tag{2.12}$$

In (2.12) if we take $f(t)=\sin_\lambda(at)$ and use (2.11) we achieve

$$\mathcal{L}_\lambda\left\{\frac{\lambda \sin_\lambda(at)}{\log(1+\lambda t)}\right\}=\int_s^\infty \frac{a}{(s-\lambda)^2+a^2}ds=\left[\tan^{-1}\left(\frac{s-\lambda}{a}\right)\right]_{s=s}^{\infty}$$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{s-\lambda}{a}\right)=\cot^{-1}\left(\frac{s-\lambda}{a}\right)$$

Or,

$$\mathcal{L}_\lambda\left\{\frac{\lambda \sin_\lambda(at)}{\log(1+\lambda t)}\right\}=\tan^{-1}\left(\frac{a}{s-\lambda}\right) \tag{2.13}$$

In our previous study [32] we have also given a result for the degenerate Laplace transform of an integral (see Theorem 2.1, p.66 in [32]) which states that:

If $\mathcal{L}_\lambda\{f(t)\}=F_\lambda(s)$ then

$$\mathcal{L}_\lambda\{f(t)\}=s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\int_0^t f(u)du\right\} \tag{2.14}$$

In (2.14) if we choose $f(t)=\dfrac{\lambda \sin_\lambda(at)}{\log(1+\lambda t)}$ we obtain, with the help of (2.13),

$$\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\int_0^t \frac{\lambda\sin_\lambda(au)}{\log(1+\lambda u)}du\right\}=\frac{1}{s}\mathcal{L}_\lambda\left\{\frac{\lambda\sin_\lambda(at)}{\log(1+\lambda t)}\right\}=\frac{1}{s}\tan^{-1}\left(\frac{a}{s-\lambda}\right)$$

which precisely proves (2.10) by keeping in mind (2.8). We note that when $a=1$, the limiting case $\lambda\to 0+$ of the Theorem 2.3 leads us to

$$\mathcal{L}\{\mathrm{Si}(t)\}=\frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right) \tag{2.15}$$

which is Problem 36, p.24 of [15].

We also mention here some interesting special cases of (2.13). With the help of (1.11) we can rewrite (2.13) as

$$\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}\frac{\lambda\sin_\lambda(at)}{\log(1+\lambda t)}dt=\tan^{-1}\left(\frac{a}{s-\lambda}\right) \tag{2.16}$$

If we take $a>0$, and put $s=\lambda+a$ in (2.16) it yields

$$\int_0^\infty (1+\lambda t)^{\frac{-(\lambda+a)}{\lambda}}\frac{\lambda\sin_\lambda(at)}{\log(1+\lambda t)}dt=\frac{\pi}{4} \tag{2.17}$$

The limiting case of (2.17) as $\lambda\to 0+$ produces

$$\int_0^\infty e^{-at}\frac{\sin(at)}{t}dt=\frac{\pi}{4} \tag{2.18}$$

which is precisely true as a special case of the following result mentioned in 3.941(1) p.497 of [5]

$$\int_0^\infty e^{-px}\frac{\sin(qx)}{x}dx=\tan^{-1}\left(\frac{q}{p}\right),\qquad p>0 \tag{2.19}$$

when $p=q=a>0$ in (2.19). We can at once see that if we put $s=\lambda+p$, (where $p>0$) and $a=q$ in (2.16) and then proceed to limits of the resulting expression as $\lambda\to 0+$ we arrive at the general result of (2.19). The limiting case of (2.16) for $s=\lambda$ when $\lambda\to 0+$ produces

$$\int_0^\infty \frac{\sin(at)}{t}dt=\frac{\pi}{2}\mathrm{sign}(a) \tag{2.20}$$

which is the result mentioned in 3.721(1), p.423 of [5] and $\mathrm{sign}(a)$ is the well known signum function (sign of a real number x) defined by (see for instance, p. xlv in [5])

$$\mathrm{sign}(x)=\begin{cases}+1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases} \tag{2.21}$$

Now we move on to define the *degenerate cosine integral* function as stated under:

Definition 2.4: The Degenerate Cosine Integral: The *degenerate cosine integral* function is defined by the following integral

$$\mathrm{Ci}_\lambda(at)=\int_t^\infty \frac{\lambda\cos_\lambda(au)}{\log(1+\lambda u)}du \tag{2.22}$$

for any $a\in\mathbb{C}$. At once we can infer from (2.22) that for $a=1$ and in the limit as $\lambda\to 0+$ (2.22) reduces to the well known (usual) cosine integral function (see, for example, (9), Appendix C, p.255, [15]) defined by

$$\mathrm{Ci}(t)=\int_t^\infty \frac{\cos u}{u}du \tag{2.23}$$

Next follows a theorem giving the degenerate Laplace transform of the degenerate cosine integral function:

Theorem 2.5: The Degenerate Laplace Transform of the Degenerate Cosine Integral:

$$\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\mathrm{Ci}_\lambda(at)\right\}=\frac{1}{2s}\log\left\{\frac{(s-\lambda)^2+a^2}{\lambda^2+a^2}\right\} \tag{2.24}$$

Proof: Let $f(t)=\mathrm{Ci}_\lambda(at)=\int_t^\infty \frac{\lambda\cos_\lambda(au)}{\log(1+\lambda u)}du$, this leads to the conclusion that $\lim_{t\to\infty} f(t)=0$. Further, as $f(t)=-\int_\infty^t \frac{\lambda\cos_\lambda(au)}{\log(1+\lambda u)}du$, from which we get on differentiation,

$$f'(t)=\frac{-\lambda\cos_\lambda(at)}{\log(1+\lambda t)}$$

i.e.,

$$\log(1+\lambda t)f'(t)=-\lambda\cos_\lambda(at) \tag{2.25}$$

Kim and Kim [30] have proved that (see (3.8) p.244 [30])

$$\mathcal{L}_\lambda\{\cos_\lambda(at)\}=\frac{s-\lambda}{(s-\lambda)^2+a^2} \tag{2.26}$$

and that (see Theorem 3.4, p.247 [30]) for $n\in\mathbb{N}$,

$$\mathcal{L}_\lambda\left\{\left[\log(1+\lambda t)\right]^n f(t)\right\}=(-1)^n\lambda^n\frac{d^n}{ds^n}\{F_\lambda(s)\} \tag{2.27}$$

where, $\mathcal{L}_\lambda\{f(t)\}=F_\lambda(s)$. Kim and Kim [30] have also established that (see (3.20), p.246 [30])

$$\mathcal{L}_\lambda\{f'(t)\}=\mathcal{L}_\lambda\left\{\frac{d}{dt}f(t)\right\}=-f(0)+s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\} \tag{2.28}$$

Now taking the degenerate Laplace transform of both sides of (2.25) we get

$$\mathcal{L}_\lambda\{\log(1+\lambda t)f'(t)\}=\mathcal{L}_\lambda\{-\lambda\cos_\lambda(at)\} \tag{2.29}$$

which by the application of (2.27) on the left side (for $n=1$) and the application of (2.26) along with the linearity property of the degenerate Laplace transform leads us to

$$(-1)\lambda\frac{d}{ds}\left[\mathcal{L}_\lambda\{f'(t)\}\right]=\frac{-\lambda(s-\lambda)}{(s-\lambda)^2+a^2} \tag{2.30}$$

The application of (2.28) on the left side of (2.30) gives

$$\frac{d}{ds}\left[-f(0)+s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}\right]=\frac{(s-\lambda)}{(s-\lambda)^2+a^2}$$

Or,

$$\frac{d}{ds}\left[s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}\right]=\frac{(s-\lambda)}{(s-\lambda)^2+a^2} \tag{2.31}$$

as $f(0)=\text{constant}$. Integration of both sides of (2.31) w.r.t. s yields

$$s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\}=\frac{1}{2}\log\left\{(s-\lambda)^2+a^2\right\}+C \qquad (2.32)$$

in which C is the constant of integration.

We have proved in our previous study [32] (see Theorem 2.5, p. 67 of [32]) the final value theorem of the degenerate Laplace transform as

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\} \qquad (2.33)$$

To evaluate the constant of integration C in (2.32), we take the limits of (2.32) as $s\to 0$ and noting that from (2.33)

$$\lim_{s\to 0} s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\}=\lim_{t\to\infty} f(t)=0, \qquad \because \text{ here } \lim_{t\to\infty} f(t)=0 \qquad (2.34)$$

we are eventually led to

$$0=\frac{1}{2}\log\left\{\lambda^2+a^2\right\}+C \qquad (2.35)$$

Substituting the value of C from (2.35) in (2.32) produces

$$\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\}=\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\operatorname{Ci}_\lambda(at)\right\}=\frac{1}{2s}\log\left\{\frac{(s-\lambda)^2+a^2}{\lambda^2+a^2}\right\} \qquad (2.36)$$

which is precisely (2.24). We mention here that when we set $a=1$ in either (2.36) or (2.24) and take the limits of either of these expressions as $\lambda\to 0+$, we arrive at the following known result (see, Problem 37, p.25 [15])

$$\mathcal{L}\left\{\operatorname{Ci}(t)\right\}=\mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\,du\right\}=\frac{1}{2s}\log\left\{s^2+1\right\} \qquad (2.37)$$

Before concluding the paper we would like to remark that we would present more results on the degenerate Laplace transforms of a function in our next communication of this series within the near future.

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